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# Coulomb torque—a general theory for electrostatic forces in many-body systems

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**Abstract**
In static experiments that comprise three conducting spheres suspended by torsion wires and held at constant electric potential, a net angular displacement about their centres has been observed. We demonstrate that the observed rotation is consistent with Coulomb's law of electrical forces complemented by Gauss' surface integrals for electrical potential. Analysis demonstrates that electrostatic torque is the result of electrostatic forces acting on an asymmetric distribution of charges residing on the surfaces of the spheres. The asymptotic value for electrostatic torque is proportional to the inverse of the fourth power of separation distance with the rotation direction, up or down taken perpendicular to a plane passing through sphere centres, given explicitly by the equation for torque. The identification of electrostatic torque prompts further analysis of models of matter at all size scales where electrostatic forces are the dominant operative force.

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## 1. Introduction

Electrostatic rotation is observed in experiments that comprise three conducting spheres fixed in space and held at constant electrical potential [1, 2]. We propose that the observed rotation is likely to be general and apply to systems of all size scales where the electrostatic force is the dominant operative force [3, 4]. This would include systems ranging in size from nuclear to macroscopic and be relevant to understanding many of the spectral, magnetic and structural properties of matter.

The purpose of this contribution is to identify the theoretical basis for the experimentally observed electrostatic rotation by evaluating the electrostatic force between $N$ conducting spheres in a three-dimensional setting. It is this last feature that makes the derivation of the many-body electrostatic force non-trivial. A way to understand the complexity of the many-body problem is to consider the action of surface charges under their mutual influence. In isolation, a charged sphere will have all its charges evenly distributed on its surface. When a second charged sphere is brought into its vicinity surface charges will instantaneously redistribute themselves under the action of their mutual influence as a function of separation distance and polar angles. We note the absence of azimuthal dependence because of the cylindrical symmetry of the system. Once a third sphere is introduced the redistributed charges in this new configuration are no longer symmetrically distributed. Indeed, the charge distribution is now generally asymmetric on the sphere surfaces with both polar and azimuthal dependence.

It is the introduction of the third sphere that breaks the cylindrical symmetry and the problem is to determine the location of surface charges on all spheres having polar and azimuthal dependence. The key advance making an explicit three-dimensional representation of the charge density distribution possible is a newly derived expansion of the potential expressed in terms of associated Legendre polynomials with complex exponentials that satisfy the boundary conditions of $N$ finite spheres [5]. We show theoretically a net electrical torque on a massive sphere with constant potential that is due to the presence of two or more spheres having constant potential. Theoretical evidence for electrostatic rotation is discussed throughout and illustrated by asymptotic analysis.

## 2. Physical and mathematical considerations

The action of charges under their mutual influence is obtained from Gauss Law that couples uniquely the applied surface potential and the geometry of the sphere configuration with the distribution and magnitude of electrical charge on the sphere surfaces [7]. In isolation, a charged sphere will have all its charges evenly distributed on its surface. Once other charged spheres are brought into its vicinity surface charges will instantaneously redistribute themselves under the action of their mutual influence. Clearly, the redistributed charges in this new sphere configuration are no longer evenly distributed on the surface. If one considers that the charge distribution will almost always be asymmetrically distributed once the third sphere is introduced, then the presence of a static moment that leads to rotation should not be a surprise.

We are interested to calculate the electrostatic force between spheres held at constant potential. The action-at-a-distance approach provides us with a straightforward way of calculating the electrostatic force with the fewest possible assumptions and it follows that the basic experimental force law proposed by Coulomb should be used directly [8]. Hence, we will not follow convention by defining an auxiliary field in order to avoid defining any vectorial quantities associated with such a field. Furthermore, we will not anticipate any symmetry of charges residing on the surfaces of the spheres because the charge distributions are uniquely obtained using Gauss' law of potentials.

The derivation of the electrostatic forces proceeds by first deriving the expansion of the surface potentials for a three-dimensional system comprising $N$ spherical conductors following the procedures outlined in earlier work [9] using a newfound expansion of the inverse of the distance that satisfies the boundary conditions for $N$ finite spheres in three-dimensional space [5]. Next, Coulomb's law is used to derive the electrostatic force and electrostatic torque for individual spheres in the three-body system. Rigor, rather than elegance, characterizes the presentation to invite investigation of our results at all levels.

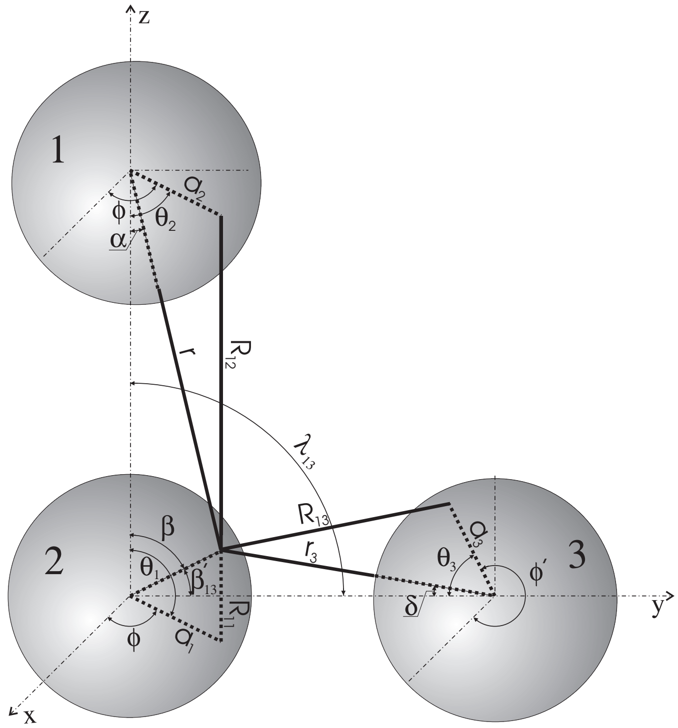

**Figure 1.** Geometric representation of three conducting spheres. Potentials, charge densities and the radii on spheres are denoted as $V_1, \sigma_1, a_1$, $V_2, \sigma_2, a_2$ and $V_3, \sigma_3, a_3$, respectively. Note, azimuth angles for spheres 1 and 2 are equal and denoted as $\phi$ while the azimuth angle for sphere 3 is $\phi_{13}$.

### *2.1. Gauss' surface potentials*

The boundary condition for the potentials on the surface of the spheres is written as [7]

$$\begin{aligned} V_1 &= K\int\frac{\mathrm{d}Q_1}{R_{11}} + K\int\frac{\mathrm{d}Q_2}{R_{12}} + K\int\frac{\mathrm{d}Q_3}{R_{13}} \\ V_2 &= K\int\frac{\mathrm{d}Q_1}{R_{21}} + K\int\frac{\mathrm{d}Q_2}{R_{22}} + K\int\frac{\mathrm{d}Q_3}{R_{23}} \\ V_3 &= K\int\frac{\mathrm{d}Q_1}{R_{31}} + K\int\frac{\mathrm{d}Q_2}{R_{32}} + K\int\frac{\mathrm{d}Q_3}{R_{33}} \end{aligned} \tag{1}$$

where $K$ is a constant of proportionality ($1/4\pi\epsilon_0$ and $\epsilon_0$ is permittivity of free space) and the length quantities $R_{ij}$ are shown in figure 1. Next we write the boundary condition on the sphere with potential $V_1$ from which the boundary conditions on $V_2$ and $V_3$ will follow by obvious interchange of symbols, i.e. cyclic permutation (123) $\rightarrow$ (231) and (231) $\rightarrow$ (312). In the spherical coordinate system $V_1$ is

$$\begin{aligned} V_1 = K\int_0^{\pi}\int_0^{2\pi} a_1^2\sin\theta_1\,\mathrm{d}\theta_1\,\mathrm{d}\phi_1\frac{\sigma_1(\theta_1,\phi_1)}{\sqrt{2a_1^2 - 2a_1^2[\cos\beta\cos\theta_1 + \sin\beta\sin\theta_1\cos(\phi-\phi_1)]}} \\ + K\int_0^{\pi}\int_0^{2\pi} a_2^2\sin\theta_2\,\mathrm{d}\theta_2\,\mathrm{d}\phi_2\frac{\sigma_2(\theta_2,\phi_2)}{\sqrt{r^2 + a_2^2 - 2a_2r[\cos\alpha\cos\theta_2 + \sin\alpha\sin\theta_2\cos(\phi-\phi_2)]}} \end{aligned}$$

$$+ K \int_0^{\pi} \int_0^{2\pi} a_3^2 \sin\theta_3 \, d\theta_3 \, d\phi_3 \frac{\sigma_3(\theta_3, \phi_3)}{\sqrt{r_3^2 + a_3^2 - 2a_3 r_3[\cos\delta\cos\theta_3 + \sin\delta\sin\theta_3\cos(\phi' - \phi_3)]}} \tag{2}$$

where

$$r = \sqrt{a_1^2 + {h_{12}}^2 - 2a_1 h_{12}\cos\beta} \qquad \cos\alpha = \frac{h_{12} - a_1\cos\beta}{\sqrt{a_1^2 + {h_{12}}^2 - 2a_1 h_{12}\cos\beta}}$$

$$r_3 = \sqrt{a_1^2 + h_{13}^2 - 2a_1 h_{13}\cos\beta'_{13}} \qquad \cos\delta = \frac{h_{13} - a_1\cos\beta'_{13}}{\sqrt{a_1^2 + h_{13}^2 - 2a_1 h_{13}\cos\beta'_{13}}}$$

and

$$\cos\beta'_{13} = \cos\lambda_{13}\cos\beta + \sin\lambda_{13}\sin\beta\cos\left(\phi - \phi_{\lambda_{13}}\right)$$

which must hold for $0 < \beta < \pi$ and $0 \leqslant \phi \leqslant 2\pi$. We note that with the introduction of the third sphere the system is no longer cylindrically symmetric and requires consideration of the azimuthal dependence. The charge density $\sigma$ must therefore be expressed in terms of both the polar angle $\theta$ and the azimuth angle $\phi$. Without loss of generality we have set the azimuth angle of sphere 2 to be the same as the azimuth angle of sphere 1. However, for the third sphere the trigonometric relation is given by $\frac{\sin\beta}{\sin(\phi_{\lambda_{13}} - \phi')} = \frac{\sin\beta'_{13}}{\sin(\phi - \phi_{\lambda_{13}})}$, which is a known relation for oblique spherical triangles [10].

It is convenient to introduce Legendre polynomials in order to express the electrostatic potential in terms of spherical harmonics. The identities used to obtain $V_1$ in equation (8) from equation (2), beginning with the first term are

$$\frac{1}{\sqrt{2a_1^2 - 2a_1^2[\cos\beta\cos\theta_1 + \sin\beta\sin\theta_1\cos(\phi - \phi_1)]}} = \frac{1}{a_1}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{(\ell - m)!}{(\ell + m)!} P_\ell^m(\cos\beta) P_\ell^m(\cos\theta_1)\, e^{im(\phi - \phi_1)} \tag{3}$$

where $P_\ell^0(x)$ are the Legendre polynomials and $P_\ell^k(x)$ are associated Legendre polynomials in the interval $-1 \leqslant x \leqslant 1$, and

$$P_\ell(x) = \sum_{j=0}^{\ell} (-1)^j \frac{(\ell + j)!}{2^j (j!)^2 (\ell - j)!}(1 - x)^j$$

$$\begin{aligned} P_\ell^k(x) &= (-1)^k (1 - x^2)^{\frac{k}{2}} \frac{d^k}{dx^k} P_\ell(x) \\ &= \sum_{j=0}^{\ell} (-1)^j \frac{(\ell + j)!}{2^j j!(\ell - j)!(j - k)!}(1 + x)^{\frac{k}{2}}(1 - x)^{j - \frac{k}{2}} \qquad |k| \leqslant \ell \end{aligned}$$

$$P_\ell^k(x) = 0 \qquad |k| > \ell$$

$$P_\ell^k(x) = (-1)^k \frac{(\ell + k)!}{(\ell - k)!} P_\ell^{-k}(x)$$

$$\int_{-1}^{1} P_\ell^k(x) P_{\ell'}^{-k}(x)\, dx = (-1)^k \frac{2}{2\ell + 1}\delta_{\ell,\ell'}$$

$$\sum_{\ell=0}^{\infty} \frac{2\ell + 1}{2} P_\ell(x) P_\ell(x') = \delta(x - x')$$

where $\delta_{\ell,\ell'}$ is the Kronecker delta and $\delta(x)$ is the Dirac delta function. The last identity is known to hold to unprecedented precision through Fredholm's formulation of the boundary conditions, which is independent of singular Kernel's appearing in the force and potential formulations developed here.

For rewriting the second term in equation (2) we use

$$\frac{1}{\sqrt{r^2+a_2^2-2ra_2[\cos\alpha\cos\theta_2+\sin\alpha\sin\theta_2\cos(\phi-\phi_2)]}}$$
$$=\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\frac{(j-m)!}{(j+m)!}\frac{a_2^j}{r^{j+1}}P_j^m(\cos\alpha)P_j^m(\cos\theta_2)\,\mathrm{e}^{\mathrm{i}m(\phi-\phi_2)}$$
$$=\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\frac{(j-m)!}{(j+m)!}\sum_{\ell=0}^{\infty}\frac{(\ell+j)!}{(\ell+m)!(j-m)!}$$
$$\times\frac{a_2^j a_1^\ell}{{h_{12}}^{\ell+j+1}}P_\ell^m(\cos\beta)P_j^m(\cos\theta_2)\,\mathrm{e}^{\mathrm{i}m(\phi-\phi_2)} \tag{4}$$

where in the last equality we have substituted for the term $\frac{P_j(\cos\alpha)}{r^{j+1}}$ the identity given by

$$\frac{P_j(\cos\alpha)}{r^{j+1}}=\frac{P_j(\cos\alpha)}{\left(a_1^2+{h_{12}}^2-2a_1h_{12}\cos\beta\right)^{\frac{(j+1)}{2}}}=\frac{(-1)^j}{j!}\frac{\partial^j}{\partial {h_{12}}^j}\frac{1}{\sqrt{a_1^2+{h_{12}}^2-2a_1h_{12}\cos\beta}}$$
$$=\frac{(-1)^j}{j!}\frac{\partial^j}{\partial {h_{12}}^j}\sum_{\ell=0}^{\infty}P_\ell(\cos\beta)\frac{a_1^\ell}{{h_{12}}^{\ell+1}}=\sum_{\ell=0}^{\infty}\frac{(\ell+j)!}{\ell!j!}\frac{a_1^\ell}{{h_{12}}^{\ell+j+1}}P_\ell(\cos\beta) \tag{5}$$

that follows by direct differentiation and substitution for the geometrical relation $h_{12}-r\cos\alpha=a_1\cos\beta$ from which

$$\frac{P_j^m(\cos\alpha)}{r^{j+1}}=\frac{(-1)^{m+j}}{(j-m)!}\frac{\partial^{j-m}}{\partial h_{12}^{j-m}}\left(\frac{1}{h_{12}^m}(\sin\beta)^m\left(\frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\right)^m\right)\frac{1}{\sqrt{a_1^2+{h_{12}}^2-2a_1h_{12}\cos\beta}}$$
$$=\sum_{\ell=0}^{\infty}\frac{(\ell+j)!}{(\ell+m)!(j-m)!}\frac{a_1^\ell}{h_{12}^{\ell+j+1}}P_\ell^m(\cos\beta) \tag{6}$$

which again may be verified by direct differentiation.

A similar procedure yields the third term:

$$\frac{1}{\sqrt{r_3^2+a_3^2-2a_3r_3[\cos\delta\cos\theta_3+\sin\delta\sin\theta_3\cos(\phi'-\phi_3)]}}$$
$$=\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\frac{(j-m)!}{(j+m)!}\sum_{\ell=0}^{\infty}\frac{(\ell+j)!}{(\ell+m)!(j-m)!}$$
$$\times\frac{a_3^j a_1^\ell}{{h_{13}}^{\ell+j+1}}P_\ell^m(\cos\beta'_{13})P_j^m(\cos\theta_3)\,\mathrm{e}^{\mathrm{i}m(\phi'-\phi_3)}. \tag{7}$$

Combining results the boundary condition for the surface potential on sphere 1 now reads

$$V_1 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Bigg[ P_\ell^m(\cos\beta) \frac{1}{a_1^{\ell+1}} \,\mathrm{e}^{\mathrm{i}m\phi} A_{\ell,m}^1 + P_\ell^m(\cos\beta)\,\mathrm{e}^{\mathrm{i}m\phi} \sum_{j=0}^{\infty} \frac{(\ell+j)!}{(\ell+m)!(j-m)!} \frac{a_1^\ell}{h_{12}^{j+\ell+1}} A_{j,m}^2$$
$$+ P_\ell^m(\cos\beta'_{13})\,\mathrm{e}^{\mathrm{i}m\phi'} \sum_{j=0}^{\infty} \frac{(\ell+j)!}{(\ell+m)!(j-m)!} \frac{a_1^\ell}{h_{13}^{j+\ell+1}} A_{j,m}^3 \Bigg] \tag{8}$$

which is the result sought after. Experimentally, the potentials on the conducting spheres are known and to complete the evaluation the coefficients in equation (8):

$$\begin{aligned}
A_{j,m}^1 &= K a_1^{j+2} \frac{(j-m)!}{(j+m)!} \int_0^\pi \int_0^{2\pi} \sin\theta_1 \,\mathrm{d}\theta_1 \,\mathrm{d}\phi_1 \sigma_1(\theta_1,\phi_1) P_j^m(\cos\theta_1)\,\mathrm{e}^{-\mathrm{i}m\phi_1} \\
A_{j,m}^2 &= K a_2^{j+2} \frac{(j-m)!}{(j+m)!} \int_0^\pi \int_0^{2\pi} \sin\theta_2 \,\mathrm{d}\theta_2 \,\mathrm{d}\phi_2 \sigma_2(\theta_2,\phi_2) P_j^m(\cos\theta_2)\,\mathrm{e}^{-\mathrm{i}m\phi_2} \\
A_{j,m}^3 &= K a_3^{j+2} \frac{(j-m)!}{(j+m)!} \int_0^\pi \int_0^{2\pi} \sin\theta_3 \,\mathrm{d}\theta_3 \,\mathrm{d}\phi_3 \sigma_3(\theta_3,\phi_3) P_j^m(\cos\theta_3)\,\mathrm{e}^{-\mathrm{i}m\phi_3}
\end{aligned} \tag{9}$$

must be evaluated in order to obtain the sought after charge densities $\sigma_n(\theta_n, \phi_n)$. With three (or more) spheres present the sphere arrangement is generally not symmetric. This leads to the third term in equation (8) that is the associated Legendre polynomials $P_l^m(\cos\beta'_{13})$ coupled to complex exponentials $\mathrm{e}^{\mathrm{i}m\phi'}$. In fact, it is at this juncture that previous attempts towards an explicit expression for the electrostatic force in a many-body system of finite spheres have faltered and will be discussed in some detail in the following section.

### 2.2. *Expansion of potentials in N body system*

In the presence of the third sphere the charge density distribution is in general asymmetric and the expansion of the potential is no longer obvious. To proceed, equation (8) is multiplied by $P_l^k(\cos\beta)\,\mathrm{e}^{\mathrm{i}k\phi}\,\mathrm{d}\cos\beta\,\mathrm{d}\phi$ and integrated using two identities. The first identity is obtained using the orthogonality relations of Legendre polynomials and reads

$$\int_{-1}^{1} \int_0^{2\pi} \mathrm{d}\cos\beta\,\mathrm{d}\phi\,\mathrm{e}^{\mathrm{i}k\phi} P_{\ell'}^k(\cos\beta) \sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\phi} P_\ell^m(\cos\beta) A_{\ell,m} = \frac{4\pi}{2\ell'+1} \delta_{\ell,\ell'} (-1)^k A_{\ell',-k}.$$

The second identity is the recently derived sum rule for the associated Legendre polynomial with complex exponentials [5]. This identity makes it possible to give the rotation required to express $P_\ell^m(\cos\beta'_{13})\,\mathrm{e}^{\mathrm{i}m\phi'}$ in terms of $\beta$ and $\phi$ (see figure 1) and thus makes it possible to expand the potential for $N$ finite spheres in three-dimensional space. This identity is

$$\int_{-1}^{1} \int_0^{2\pi} \mathrm{d}\cos\beta\,\mathrm{d}\phi\,\mathrm{e}^{\mathrm{i}k\phi} P_{\ell'}^k(\cos\beta) \sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\phi'} P_\ell^m(\cos\beta'_{13}) B_{j,m}$$
$$= \frac{4\pi}{2\ell'+1} \delta_{\ell,\ell'} \sum_{m=-\ell}^{\ell} B_{j,m}\,\mathrm{e}^{\mathrm{i}(m+k)\phi_{\lambda_{13}}} (-1)^{m+\ell+k} g_{\ell,k}^m(-\cos\lambda_{13}) \qquad k \geqslant 0 \tag{10}$$

where

$$g_{\ell,k}^m(x) = \sum_{n=0}^{\ell} \frac{(-1)^n(\ell+n)!}{2^n(n-m)!(\ell-n)!(n-k)!} (1+x)^{\frac{k+m}{2}} (1-x)^{n-\frac{k+m}{2}}$$
$$\text{for} \quad |k| \leqslant \ell \quad |m| \leqslant \ell.$$

After making the appropriate substitutions the boundary condition on sphere 1 can now be written as

$$
\begin{aligned}
V_1\delta_{k,0}\,\delta_{\ell,0} = {}& (-1)^k \frac{1}{a_1^{\ell+1}} A^1_{\ell,-k} + (-1)^k \sum_{j=0}^{\infty} \frac{(\ell+j)!}{(\ell-k)!(j+k)!} \frac{a_1^\ell}{h_{12}^{j+\ell+1}} A^2_{j,-k} \\
& + \sum_{j=0}^{\infty} (\ell+j)! \frac{a_1^\ell}{h_{13}^{j+\ell+1}} \sum_{m=-\ell}^{\ell} \frac{g^m_{\ell,k}(-\cos\lambda_{13})}{(\ell+m)!(j-m)!} (-1)^{m+\ell+k}\, \mathrm{e}^{\mathrm{i}(m+k)\phi_{\lambda_{13}}} A^3_{j,m} \\
& \text{for} \quad k \geqslant 0.
\end{aligned} \tag{11}
$$

Further, generalizing equation (11) for a system containing $N$ spheres the boundary condition on sphere 1 can be written as

$$
\begin{aligned}
V_1\delta_{k,0}\,\delta_{\ell,0} = {}& (-1)^k \frac{1}{a_1^{\ell+1}} A^1_{\ell,-k} + (-1)^k \sum_{j=0}^{\infty} \frac{(\ell+j)!}{(\ell-k)!(j+k)!} \frac{a_1^\ell}{h_{12}^{j+\ell+1}} A^2_{j,-k} \\
& \times \sum_{n=3}^{N} \sum_{j=0}^{\infty} (\ell+j)! \frac{a_1^\ell}{h_{1n}^{j+\ell+1}} \sum_{m=-\ell}^{\ell} \frac{g^m_{\ell,k}(-\cos\lambda_{1n})}{(\ell+m)!(j-m)!} (-1)^{m+\ell+k}\, \mathrm{e}^{\mathrm{i}(m+k)\phi_{\lambda_{1n}}} A^n_{j,m} \\
& \text{for} \quad k \geqslant 0.
\end{aligned} \tag{12}
$$

The complex conjugate of above equation is equivalent to the case of $k \leqslant 0$.

Once the above equation is written in full form then the boundary conditions for $V_2, V_3, \ldots, V_N$ can be obtained by cyclic permutation. The generalized expression for the coefficients in equation (12) for $N$ spheres reads

$$
\begin{aligned}
A^n_{j,m} &= K a_n^j (-1)^m \int \mathrm{d}Q_n\, P_j^{-m}(\cos\theta_n)\, \mathrm{e}^{-\mathrm{i}m\phi_n} \\
&= K a_n^j (-1)^m \int_0^\pi \int_0^{2\pi} \sin\theta_n\, \mathrm{d}\theta_n\, \mathrm{d}\phi_n a_n^2 \sigma_n(\theta_n,\phi_n)\, P_j^{-m}(\cos\theta_n)\, \mathrm{e}^{-\mathrm{i}m\phi_n} \qquad |m| \leqslant j
\end{aligned} \tag{13}
$$

from which the charge densities $\sigma_n(\theta_n, \phi_n)$ are obtained.

It was the newfound sum rule for the associated Legendre polynomial with complex exponentials which made possible the above simple equation for the boundary conditions. The important result is that the expression for constant potential boundary condition equation (12) accounts for all contributions of charge, both polar and azimuthal, stemming from all surfaces without assuming symmetry.

### *2.3. Coulomb force*

The force on sphere 1 due to charges on spheres 2 and 3 is

$$
\vec{F}_1 = \vec{F}_{21} + \vec{F}_{31} = -\vec{F}_{12} - \vec{F}_{13}. \tag{14}
$$

Taking equation (14) to be valid only at the surface of the sphere, noting that the sharp discontinuity at the sphere surface renders the differentiation of the potential to be ill-defined, we use Coulomb's law directly to avoid making any assumption of auxiliary fields:

$$
\begin{aligned}
\vec{F}_1 &= K \int \mathrm{d}Q_1\, \mathrm{d}Q_2 \frac{\vec{R}_{12}}{R_{12}^3} + K \int \mathrm{d}Q_1\, \mathrm{d}Q_3 \frac{\vec{R}_{13}}{R_{13}^3} \\
&= K \int \mathrm{d}Q_1\, \mathrm{d}Q_2 \left(\vec{\nabla}_{21} \frac{1}{R_{12}}\right) + K \int \mathrm{d}Q_1\, \mathrm{d}Q_3 \left(\vec{\nabla}_{31} \frac{1}{R_{13}}\right)
\end{aligned} \tag{15}
$$

where the integration is over spheres $\int \mathrm{d}Q_1 \mathrm{d}Q_2 = \int_0^{\pi}\int_0^{2\pi} a_1^2 \sin\theta_1 \,\mathrm{d}\theta_1\,\mathrm{d}\phi_1 \int_0^{\pi}\int_0^{2\pi} a_2^2 \sin\theta_2 \,\mathrm{d}\theta_2\,\mathrm{d}\phi_2$ and the gradients $\vec{\nabla}_{21}$ and $\vec{\nabla}_{31}$ are taken from the translated points which is at the centre of the spheres 2 and 3, respectively.

It is convenient to proceed by first examining the force on sphere 1 due to sphere 2 as the contributions from sphere 3 (and others) are readily obtained by interchange of indices. We begin by writing the gradient $\vec{\nabla}_{21}$ in spherical polar coordinates:

$$\vec{\nabla}_{21} = \hat{r}\frac{\partial}{\partial r} + \hat{\alpha}\frac{1}{r}\frac{\partial}{\partial \alpha} + \hat{\phi}\frac{1}{r\sin\alpha}\frac{\partial}{\partial\phi} \tag{16}$$

from which we can express $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$, also in spherical coordinates, by equating $\nabla_{xyz}$ with $\nabla_{r\theta\phi}$:

$$\begin{aligned}
\frac{\partial}{\partial x} &= \sin\theta_1\cos\phi_1\frac{\partial}{\partial a_1} + \frac{\cos\theta_1\cos\phi_1}{a_1}\frac{\partial}{\partial\theta_1} - \frac{\sin\phi_1}{a_1\sin\theta_1}\frac{\partial}{\partial\phi_1}\\
\frac{\partial}{\partial y} &= \sin\theta_1\sin\phi_1\frac{\partial}{\partial a_1} + \frac{\cos\theta_1\sin\phi_1}{a_1}\frac{\partial}{\partial\theta_1} + \frac{\cos\phi_1}{a_1\sin\theta_1}\frac{\partial}{\partial\phi_1}\\
\frac{\partial}{\partial z} &= \cos\theta_1\frac{\partial}{\partial a_1} - \frac{\sin\theta_1}{a_1}\frac{\partial}{\partial\theta_1}
\end{aligned}$$

where the radius of sphere 1 is $\vec{a}_1 = \hat{x}\sin\theta_1\cos\phi_1 + \hat{y}\sin\theta_1\sin\phi_1 + \hat{z}\cos\theta_1$. To facilitate the derivation we introduce the operators $p_+$ and $p_-$ defined according to

$$p_+ = p_x + \mathrm{i}p_y \qquad p_- = p_x - \mathrm{i}p_y$$

for $\vec{p} = \vec{\nabla}_{12} = -\vec{\nabla}_{21}$ which allows the gradient $\vec{\nabla}_{12}$ to be expressed as

$$\begin{aligned}
\vec{\nabla}_{12} = {}& \frac{1}{2}(\hat{x} - \mathrm{i}\hat{y})\,\mathrm{e}^{\mathrm{i}\phi_1}\left(\sin\theta_1\frac{\partial}{\partial a_1} + \frac{\cos\theta_1}{a_1}\frac{\partial}{\partial\theta_1} + \mathrm{i}\frac{1}{a_1\sin\theta_1}\frac{\partial}{\partial\phi_1}\right)\\
&+ \frac{1}{2}(\hat{x} + \mathrm{i}\hat{y})\,\mathrm{e}^{\mathrm{i}\phi_1}\left(\sin\theta_1\frac{\partial}{\partial a_1} + \frac{\cos\theta_1}{a_1}\frac{\partial}{\partial\theta_1} - \mathrm{i}\frac{1}{a_1\sin\theta_1}\frac{\partial}{\partial\phi_1}\right)\\
&+ \hat{z}\left(\cos\theta_1\frac{\partial}{\partial a_1} - \frac{\sin\theta_1}{a_1}\frac{\partial}{\partial\theta_1}\right)
\end{aligned} \tag{17}$$

where we identify

$$\begin{aligned}
p_+ &= \mathrm{e}^{\mathrm{i}\phi_1}\left(\sin\theta_1\frac{\partial}{\partial a_1} + \frac{\cos\theta_1}{a_1}\frac{\partial}{\partial\theta_1} + \mathrm{i}\frac{1}{a_1\sin\theta_1}\frac{\partial}{\partial\phi_1}\right)\\
p_- &= \mathrm{e}^{-\mathrm{i}\phi_1}\left(\sin\theta_1\frac{\partial}{\partial a_1} + \frac{\cos\theta_1}{a_1}\frac{\partial}{\partial\theta_1} - \mathrm{i}\frac{1}{a_1\sin\theta_1}\frac{\partial}{\partial\phi_1}\right)\\
p_z &= \left(\cos\theta_1\frac{\partial}{\partial a_1} - \frac{\sin\theta_1}{a_1}\frac{\partial}{\partial\theta_1}\right)
\end{aligned}$$

and

$$\vec{\nabla}_{12} = \tfrac{1}{2}(\hat{x} - \mathrm{i}\hat{y})p_+ + \tfrac{1}{2}(\hat{x} + \mathrm{i}\hat{y})p_- + \hat{z}p_z. \tag{18}$$

Similarly, given the definition of the operator Coulomb's law for the electrostatic force on sphere 1 due to sphere 2 can be written as

$$\begin{aligned}
\vec{F}_{12} = {}& \frac{K}{2}(\hat{x} - \mathrm{i}\hat{y})\int \mathrm{d}Q_1\,\mathrm{d}Q_2\left(p_+\frac{1}{R_{12}}\right) + \frac{K}{2}(\hat{x} + \mathrm{i}\hat{y})\int \mathrm{d}Q_1\,\mathrm{d}Q_2\left(p_-\frac{1}{R_{12}}\right)\\
&+ K\hat{z}\int \mathrm{d}Q_1\,\mathrm{d}Q_2\left(p_z\frac{1}{R_{12}}\right).
\end{aligned} \tag{19}$$

Substitution of the inverse of the separation distance $1/R_{12}$ into the first term of equation (19) yields

$$\int \mathrm{d}Q_1\,\mathrm{d}Q_2\,p_+\frac{1}{R_{12}} = \int \mathrm{d}Q_1 \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\ell=0}^{\infty}\frac{(\ell+j)!}{(\ell+m)!(j-m)!}\frac{A_{j,m}^2}{K{h_{12}}^{\ell+j+1}}\,p_+a_1^\ell P_\ell^m(\cos\theta_1)\,\mathrm{e}^{\mathrm{i}m\phi_1}. \tag{20}$$

Next, we will use the following recursion relations for associated Legendre polynomials:

$$p_+a_1^\ell\,\mathrm{e}^{\mathrm{i}m\phi_1}P_\ell^m(\cos\theta_1) = a_1^{\ell-1}\,\mathrm{e}^{\mathrm{i}(m+1)\phi_1}P_{\ell-1}^{m+1}(\cos\theta_1)$$
$$p_z a_1^\ell\,\mathrm{e}^{\mathrm{i}m\phi_1}P_\ell^m(\cos\theta_1) = a_1^{\ell-1}(\ell+m)\,\mathrm{e}^{\mathrm{i}m\phi_1}P_{\ell-1}^{m}(\cos\theta_1)$$

and substitute back into equation (20) to obtain

$$\begin{aligned}\int \mathrm{d}Q_1\,\mathrm{d}Q_2\,p_+\frac{1}{R_{12}} &\\ &= \frac{1}{a_1}\int \mathrm{d}Q_1\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\ell=1}^{\infty}\frac{(\ell+j)!}{(\ell+m)!(j-m)!}\frac{A_{j,m}^2a_1^\ell}{K{h_{12}}^{\ell+j+1}}P_{\ell-1}^{m+1}(\cos\theta_1)\,\mathrm{e}^{\mathrm{i}(m+1)\phi_1}\\ &= \int \mathrm{d}Q_1\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\ell=1}^{\infty}\frac{(\ell+j)!}{(\ell-m)!(j+m)!}\frac{A_{j,-m}^2a_1^{\ell-1}}{K{h_{12}}^{\ell+j+1}}P_{\ell-1}^{-m+1}(\cos\theta_1)\,\mathrm{e}^{-\mathrm{i}(m-1)\phi_1}\end{aligned} \tag{21}$$

that is

$$\int \mathrm{d}Q_1\,\mathrm{d}Q_2\,p_+\frac{1}{R_{12}} = \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\ell=1}^{\infty}(-1)^{m+1}\frac{(\ell+j)!}{(\ell-m)!(j+m)!}\frac{A_{j,-m}^2A_{\ell-1,m-1}^1}{K^2{h_{12}}^{\ell+j+1}}. \tag{22}$$

Similarly, the second term reads

$$\int \mathrm{d}Q_1\,\mathrm{d}Q_2\,p_-\frac{1}{R_{12}} = \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\ell=1}^{\infty}(-1)^{m+1}\frac{(\ell+j)!}{(\ell-m)!(j+m)!}\frac{A_{j,-m}^{2*}A_{\ell-1,m-1}^{1*}}{K^2{h_{12}}^{\ell+j+1}} \tag{23}$$

where $*$ is used to define complex conjugation. And, the third term is written as

$$\int \mathrm{d}Q_1\,\mathrm{d}Q_2\,p_z\frac{1}{R_{12}} = \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\ell=1}^{\infty}(-1)^{m}\frac{(\ell-m)(\ell+j)!}{(\ell-m)!(j+m)!}\frac{A_{j,-m}^2A_{\ell-1,m}^1}{K^2{h_{12}}^{\ell+j+1}}. \tag{24}$$

Substitution of equations (22)–(24) into equation (19) yields the sought after expression for the Coulomb force relative to the centre of sphere 1 due to sphere 2:

$$\begin{aligned}\vec{F}_{21} = -\vec{F}_{12} = {}&-\hat{x}\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\ell=1}^{\infty}(-1)^{m+1}\frac{(\ell+j)!}{(\ell-m)!(j+m)!}\frac{A_{j,-m}^2A_{\ell-1,m-1}^1}{K{h_{12}}^{\ell+j+1}}\\ &-\hat{z}\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\ell=1}^{\infty}(-1)^{m}\frac{(\ell-m)(\ell+j)!}{(\ell-m)!(j+m)!}\frac{A_{j,-m}^2A_{\ell-1,m}^1}{K{h_{12}}^{\ell+j+1}}.\end{aligned} \tag{25}$$

It follows that the force on sphere 1 due to charges on sphere 3 is obtained from the above equations by obvious interchange of indices 1–3.

The net force in a closed system of $i$ spheres follows by vector addition of *all* forces acting in the system such that:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_i = 0 \tag{26}$$

in accordance with the conservation laws. Experimentally, the constant potential limit describes the physical situation of spheres connected to a voltage power supply. It should thus be possible to verify the electrostatic force experimentally for selected many-body configurations using methods developed in previous work [11].

### *2.4. Coulomb torque*

The Coulomb torque is evaluated in an analogous fashion. The torque relative to the centre of sphere 1 due to spheres 2 and 3 follow from the expression for the Coulomb force:

$$\vec{T}_1 = -\vec{T}_{12} - \vec{T}_{13}. \tag{27}$$

Again, we will use Coulomb's law directly to write

$$\begin{aligned}\vec{T}_1 &= K \int \mathrm{d}Q_1\,\mathrm{d}Q_2 \left(\vec{a}_1 \times \frac{\vec{R}_{12}}{R_{12}^3}\right) + K \int \mathrm{d}Q_1\,\mathrm{d}Q_3 \left(\vec{a}_1 \times \frac{\vec{R}_{13}}{R_{13}^3}\right) \\ &= K \int \mathrm{d}Q_1\,\mathrm{d}Q_2 \left(\vec{a}_1 \times \vec{\nabla}_{21} \frac{1}{R_{12}}\right) + K \int \mathrm{d}Q_1\,\mathrm{d}Q_3 \left(\vec{a}_1 \times \vec{\nabla}_{31} \frac{1}{R_{13}}\right)\end{aligned} \tag{28}$$

where $a_1$, $\int \mathrm{d}Q_1\,\mathrm{d}Q_2$ and $\nabla_{21}$ and $\nabla_{31}$ have already been defined. For deriving the expression for Coulomb torque we introduce the operators $L_+$ and $L_-$ defined according to

$$L_+ = L_x + \mathrm{i}L_y \qquad L_- = L_x - \mathrm{id}L_y$$

for $\vec{L}(\theta_1, \phi_1) = a_1 \times \vec{p}$ where $\vec{p} = \vec{\nabla}_{12}$, which allows the operator $\vec{L}(\theta_1, \phi_1)$ to be expressed as

$$\begin{aligned}\vec{L}(\theta_1, \phi_1) = \frac{1}{2}(\hat{y} + \mathrm{i}\hat{x})\,\mathrm{e}^{\mathrm{i}\phi_1} \left(\frac{\partial}{\partial\theta_1} + \mathrm{i}\cot\theta_1 \frac{\partial}{\partial\phi_1}\right) \\ + \frac{1}{2}(\hat{y} - \mathrm{i}\hat{x})\,\mathrm{e}^{-\mathrm{i}\phi_1} \left(\frac{\partial}{\partial\theta_1} - \mathrm{i}\cot\theta_1 \frac{\partial}{\partial\phi_1}\right) + \hat{z}\frac{\partial}{\partial\phi_1}\end{aligned} \tag{29}$$

where we identify

$$L_+ = \mathrm{e}^{\mathrm{i}\phi_1} \left(\frac{\partial}{\partial\theta_1} + \mathrm{i}\cot\theta_1 \frac{\partial}{\partial\phi_1}\right) \qquad L_- = \mathrm{e}^{-\mathrm{i}\phi_1} \left(\frac{\partial}{\partial\theta_1} - \mathrm{i}\cot\theta_1 \frac{\partial}{\partial\phi_1}\right) \qquad L_z = \frac{\partial}{\partial\phi_1} \tag{30}$$

and we obtain

$$\vec{L}(\theta_1, \phi_1) = \tfrac{1}{2}(\hat{y} + \mathrm{i}\hat{x})L_+ + \tfrac{1}{2}(\hat{y} - \mathrm{i}\hat{x})L_- + \hat{z}L_z. \tag{31}$$

Given the definition of the operator $\vec{L}$ the expression for Coulomb torque on sphere 1 due to sphere 2 can be written as

$$\begin{aligned}\vec{T}_{12} = \frac{K}{2}(\hat{y} + \mathrm{i}\hat{x}) \int \mathrm{d}Q_1\,\mathrm{d}Q_2\, L_+ \frac{1}{R_{12}} + \frac{K}{2}(\hat{y} - \mathrm{i}\hat{x}) \int \mathrm{d}Q_1\,\mathrm{d}Q_2\, L_- \frac{1}{R_{12}} \\ + K\hat{z} \int \mathrm{d}Q_1\,\mathrm{d}Q_2\, L_z \frac{1}{R_{12}}.\end{aligned} \tag{32}$$

We begin by evaluating the first term of the above expression by substituting the expression for the inverse of the separation distance $1/R_{12}$ which yields

$$\begin{aligned}\int \mathrm{d}Q_1\,\mathrm{d}Q_2\, L_+ \frac{1}{R_{12}} = \int \mathrm{d}Q_1 \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \sum_{\ell=0}^{\infty} \frac{(\ell + j)!}{(\ell + m)!(j - m)!} \\ \times \frac{A_{j,m}^2}{K h_{12}{}^{\ell+j+1}} L_+ a_1^\ell P_\ell^m(\cos\theta_1)\,\mathrm{e}^{\mathrm{i}m\phi_1}.\end{aligned} \tag{33}$$

Again, we will use the same recursion relations for associated Legendre polynomials as before but now written in terms of the operator $L$:

$$L_{+} a_1^{\ell}\, \mathrm{e}^{\mathrm{i}m\phi_1} P_{\ell}^{m}(\cos\theta_1) = a_1^{\ell}\, \mathrm{e}^{\mathrm{i}(m+1)\phi_1} P_{\ell}^{m+1}(\cos\theta_1)$$

$$L_{z} a_1^{\ell}\, \mathrm{e}^{\mathrm{i}m\phi_1} P_{\ell}^{m}(\cos\theta_1) = a_1^{\ell}(\mathrm{i}m)\, \mathrm{e}^{\mathrm{i}m\phi_1} P_{\ell}^{m}(\cos\theta_1)$$

and substituting back into equation (33) to obtain

$$\begin{aligned}\int \mathrm{d}Q_1\, \mathrm{d}Q_2\, L_{+}\frac{1}{R_{12}} &= \int \mathrm{d}Q_1 \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\ell=1}^{\infty} \frac{(\ell+j)!}{(\ell+m)!(j-m)!}\frac{A_{j,m}^{2} a_1^{\ell}}{K h_{12}{}^{\ell+j+1}} P_{\ell}^{m+1}(\cos\theta_1)\, \mathrm{e}^{\mathrm{i}(m+1)\phi_1} \\ &= \int \mathrm{d}Q_1 \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\ell=1}^{\infty} \frac{(\ell+j)!}{(\ell-m)!(j+m)!}\frac{A_{j,-m}^{2} a_1^{\ell}}{K h_{12}{}^{\ell+j+1}} P_{\ell}^{-m+1}(\cos\theta_1)\, \mathrm{e}^{-\mathrm{i}(m-1)\phi_1}\end{aligned} \tag{34}$$

that is

$$\int \mathrm{d}Q_1\, \mathrm{d}Q_2\, L_{+}\frac{1}{R_{12}} = \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\ell=1}^{\infty} (-1)^{m+1} \frac{(\ell+j)!}{(\ell-m)!(j+m)!}\frac{A_{j,-m}^{2} A_{\ell,m-1}^{1}}{K^2 h_{12}{}^{\ell+j+1}}. \tag{35}$$

Similarly, the second term reads (taking the complex conjugate of the above)

$$\int \mathrm{d}Q_1\, \mathrm{d}Q_2\, L_{-}\frac{1}{R_{12}} = \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\ell=1}^{\infty} (-1)^{m+1} \frac{(\ell+j)!}{(\ell-m)!(j+m)!}\frac{A_{j,-m}^{*2} A_{\ell,m-1}^{*1}}{K^2 h_{12}{}^{\ell+j+1}} \tag{36}$$

and, the third term is

$$\int \mathrm{d}Q_1\, \mathrm{d}Q_2\, L_{z}(\phi_1)\frac{1}{R_{12}} = \mathrm{i}\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\ell=1}^{\infty} m(-1)^{m} \frac{(\ell+j)!}{(\ell-m)!(j+m)!}\frac{A_{j,-m}^{2} A_{\ell,m}^{1}}{K^2 h_{12}{}^{\ell+j+1}}. \tag{37}$$

Combining terms yields the sought after expression for the Coulomb torque relative to the centre of sphere 1 due to sphere 2:

$$-\vec{T}_{12} = -\hat{y}\frac{1}{K}\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\ell=1}^{\infty} (-1)^{m+1} \frac{(\ell+j)!}{(\ell-m)!(j+m)!}\frac{A_{\ell,m-1}^{1} A_{j,-m}^{2}}{K h_{12}{}^{\ell+j+1}}. \tag{38}$$

It follows that the torque on sphere 1 due to charges on sphere 3 is obtained from the above equations by obvious interchange of indices 1, 2 and 3 observing the geometric relations of oblique spherical triangles. The net torque in a closed system of $i$ spheres follows by vector addition of *all* torques acting in the system such that

$$\frac{\vec{T}_1}{a_1} + \frac{\vec{T}_2}{a_2} + \frac{\vec{T}_3}{a_3} = 0 \tag{39}$$

in accordance with the conservation laws.

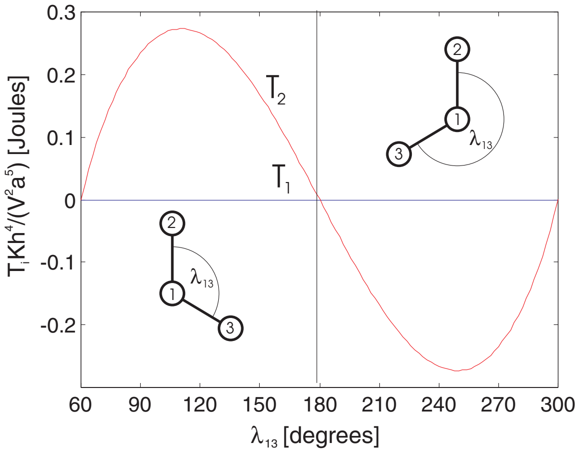


**Figure 2.** Asymptotic values for electrostatic torque for three spheres of equal size and surface potential. The rotation direction is given explicitly by the equation for torque. The direction is either positive or negative taken perpendicular to the plane passing through the sphere centres. Rotation direction for sphere 2 is positive when sphere 3 is positioned such that $60^\circ < \lambda_{13} < 180^\circ$ and negative when positioned such that $180^\circ < \lambda_{13} < 300^\circ$.

The asymptotic expressions of the Coulomb torque for three spheres are

$$\begin{aligned}
\vec{T}_{1\infty} &= -\hat{y}\frac{1}{K}\left(\frac{A^1_{1,1}A^2_{0,0} - A^1_{0,0}A^2_{1,1}}{h^2_{12}} - \frac{A^1_{1,1}A^3_{0,0} - A^1_{0,0}A^3_{1,1}}{h^2_{13}}\right)\\
\vec{T}_{2\infty} &= -\hat{y}\frac{1}{K}\left(\frac{A^2_{1,1}A^3_{0,0} - A^2_{0,0}A^3_{1,1}}{h^2_{23}} - \frac{A^2_{1,1}A^1_{0,0} - A^2_{0,0}A^1_{1,1}}{h^2_{12}}\right)\\
\vec{T}_{3\infty} &= -\hat{y}\frac{1}{K}\left(\frac{A^3_{1,1}A^1_{0,0} - A^3_{0,0}A^1_{1,1}}{h^2_{13}} - \frac{A^3_{1,1}A^1_{0,0} - A^3_{0,0}A^2_{1,1}}{h^2_{23}}\right)
\end{aligned} \tag{40}$$

where $\lambda_{13} + \lambda_{21} + \lambda_{32} = \pi$ and $\frac{\sin\lambda_{13}}{h_{23}} = \frac{\sin\lambda_{21}}{h_{31}} = \frac{\sin\lambda_{32}}{h_{12}}$, and where the first three coefficients $A^n_{j,m}$ in equation (38) are obtained by cyclic permutation:

$$A^1_{0,0} = a_1V_1 = KQ_{1\infty} \qquad A^2_{0,0} = a_2V_2 = KQ_{2\infty} \qquad A^3_{0,0} = a_3V_3 = KQ_{3\infty}$$

$$A^1_{1,0} = -a_1\cos^2\frac{\lambda_{13}}{2}\left(V_2a_2\frac{a_1^2}{h^2_{12}} + V_3a_3\frac{a_1^2}{h^2_{13}}\right) \qquad A^2_{1,0} = -a_2\cos^2\frac{\lambda_{21}}{2}\left(V_3a_3\frac{a_2^2}{h^2_{23}} + V_1a_1\frac{a_2^2}{h^2_{21}}\right)$$

$$A^3_{1,0} = -a_3\cos^2\frac{\lambda_{32}}{2}\left(V_1a_1\frac{a_3^2}{h^2_{31}} + V_2a_2\frac{a_3^2}{h^2_{32}}\right) \qquad A^1_{1,-1} = a_1\frac{\sin\lambda_{13}}{2}\left(V_2a_2\frac{a_1^2}{h^2_{12}} + V_3a_3\frac{a_1^2}{h^2_{13}}\right)$$

$$A^2_{1,-1} = a_2\frac{\sin\lambda_{21}}{2}\left(V_3a_3\frac{a_2^2}{h^2_{23}} + V_1a_1\frac{a_2^2}{h^2_{21}}\right) \qquad A^3_{1,-1} = a_3\frac{\sin\lambda_{32}}{2}\left(V_1a_1\frac{a_3^2}{h^2_{31}} + V_2a_2\frac{a_3^2}{h^2_{32}}\right).$$

The asymptotic value of the Coulomb torque is proportional to the inverse of the fourth power of separation distance and is plotted in figure 2 for three spheres with equal radii $a_1 = a_2 = a_3 = a$. The spheres are located such that spheres 2 and 3 lie on the perimeter of a circle with its centre at sphere 1.

We find that torque is generally present for all sphere arrangements except when the centres of spheres align along a common axis (cylindrical symmetry—no azimuthal dependence) and when three spheres are arranged in an equilateral triangle: a prediction that is in agreement with experimental observations [1–4]. The rotation direction is given explicitly by the equation for torque and is either up or down taken perpendicular to the plane passing through the sphere centres. In figure 2 we find that the spin direction for sphere 2 is positive when $60^\circ < \lambda_{13} < 180^\circ$ but negative for $180^\circ < \lambda_{13} < 300^\circ$. For sphere 3 the spin direction is opposite to that of sphere 2.

## 3. Discussion

We have shown the existence of electrostatic torque for conducting spheres to be a natural consequence of electrical action-at-a-distance force. Theory predicts that electrostatic rotation is the direct consequence of the Coulomb force acting on an asymmetric distribution of charges residing on the surface of the conductors. The mathematical structure of electrostatic spin is shown to be the consequence of the expansion of the potentials for *N* spheres which forces the dependence of both polar and azimuthal contributions to the surface charge distribution. It is the presence of the third body that forces rotation. The identification of electrostatic rotation was prompted by experimental observations and confirmed theoretically from fundamental laws of electrostatics notwithstanding the postulated direction of an auxiliary electric field [12].

The first half of the nineteenth century was marked by the discovery of a variety of new phenomena in electricity and magnetism and the general task to which scientists then addressed themselves was to develop a unified theory of electromagnetism [13]. At the time, the theory of continuum mechanics and associated differential equations were available but the motivation for applying these to electricity and magnetism was not yet apparent. Thomson's initial work in this area was in electrostatics and was guided by certain analogies between electrostatics as treated by Laplace and Poisson and heatflow as treated by Fourier, which resulted in a mathematical approach to electrostatics that emphasized the spatial distribution and geometrical relationships of electrical forces [13, 14]. Even though the concept of an electrostatic potential was justified for integrating magnetic and electrical phenomena the experimental verification of postulated electrical quantities including the electrostatic force received much less attention [15, 16]. Indeed, the primary motivation at the time was the determination of the ratio of the electromagnetic to the electrostatic charge unit, an essential quantity for predicting the signalling performance of long cables, a problem of great practical importance at the time [17]. In fact, it is plausible that the success of the new field theory [18] for solving important technological problems at the time made it less important to evaluate its limitations.

The identification of electrostatic rotation was prompted by experimental observations and confirmed theoretically from fundamental laws of electrostatics notwithstanding the postulated direction of an auxiliary electric field [12]. It should be noted that the findings presented here do not conflict with previous work. Theoretical evidence for Coulomb torque was obtained from the classical definition of the static moment of force by considering the interaction of three charged spheres arranged in different configurations that yield either symmetric or asymmetric surface charge distributions. Previously, only cylindrically symmetric interactions that comprise two surfaces, spheres [19] or discs [17], have been investigated experimentally and theoretically. As it turns out either method of analysis, field theory or the method described here (see also [9]), yields the same result for two interacting objects in the absence of azimuthal dependence. Once, the third sphere is introduced the electrical charges instantaneously redistribute themselves on the sphere surfaces under the action of their mutual influence. Generally, the electrical charges are asymmetrically distributed on the sphere surfaces and both polar and azimuthal dependence must be accounted for. We find that electrostatic rotation is the direct consequence of the Coulomb force acting on an asymmetric distribution of surface charges. It follows that if the direction of the electric field is taken to be normal to the equipotential surface this introduces a restrictive assumption *a priori* that does not permit electrostatic rotation contrary to experimental observations [1–4] and theoretical analysis presented here.

## 4. Conclusion

The experimental observation of a rotational force in an electrostatic system of three spherical conductors testifies to the existence of a Coulomb torque. The experimentally observed rotation cannot be explained using the conventional assumption of an electric field directed outward normal to an equipotential surface—an assumption that automatically precludes tangential forces. However, the observed rotation is correctly predicted by an explicit solution to the electrostatic problem given Gauss' definition of the boundary conditions on the spheres and Coulomb's law of the electrostatic force without invoking any approximations or simplifications with respect to vectorial quantities of an auxiliary electric field. The discovery of electrostatic rotation invites investigation of systems of all size-scales where the electrostatic force is the dominant operative force. This would include materials at the atomic and molecular scales and be relevant to understanding their spectral and structural properties.

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